

IFUP-TH 47/98

The Nonlinear Schrödinger Equation on the Half Line

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Abstract

The nonlinear Schrödinger equation on the half line with mixed boundary condition is investigated. After a brief introduction to the corresponding classical boundary value problem, the exact second quantized solution of the system is constructed. The construction is based on a new algebraic structure, which is called in what follows boundary algebra and which substitutes, in the presence of boundaries, the familiar Zamolodchikov-Faddeev algebra. The fundamental quantum field theory properties of the solution are established and discussed in detail. The relative scattering operator is derived in the Haag-Ruelle framework, suitably generalized to the case of broken translation invariance in space.

October 1998

I. INTRODUCTION

The general interest in quantization on the half line $\mathbf{R}_+ = \{x \in \mathbf{R} : x > 0\}$ stems from the recently growing number of applications in different physical areas, including open string theory, dissipative quantum mechanics and quantum impurity problems. In the last few years, important progress has been made in this subject by means of conformal field theory. Focusing on the nonlinear Schrödinger (NLS) model on \mathbf{R}_+ , in the present paper we explore the possibility to employ integrability.

Let us recall that when considered on the whole line \mathbf{R} , the NLS model represents one of the most extensively studied nonrelativistic integrable systems (see e.g. Ref. 1). The corresponding equation of motion is

$$(i\partial_t + \partial_x^2)\Phi(t, x) = 2g |\Phi(t, x)|^2 \Phi(t, x) \quad , \quad (1.1)$$

where $\Phi(t, x)$ is a classical complex field. The model on the half line is obtained restricting Eq.(1.1) on \mathbf{R}_+ , supplemented with the boundary condition

$$\lim_{x \downarrow 0} (\partial_x - \eta) \Phi(t, x) = 0 \quad . \quad (1.2)$$

Here η is a dimensionful parameter of the theory. For $\eta = 0$ and in the limit $\eta \rightarrow \infty$ one recovers from Eq.(1.2) the familiar Neumann and Dirichlet boundary conditions respectively. To our knowledge, the boundary value problem (1.1,2) has been first investigated by Sklyanin² and Fokas³, who have shown that the integrability, which holds for the system on the whole line, persists also on the half line. Our main goal below will be to construct the exact second quantized solution of Eqs.(1.1,2), in the case $g \geq 0, \eta \geq 0$. Concretely, this means:

1. To construct a Hilbert space $\mathcal{H}_{g,\eta}$ describing the states of the system;
2. To define on an appropriate dense domain in $\mathcal{H}_{g,\eta}$ an operator valued distribution $\Phi(t, x)$, $x > 0$, satisfying, in a sense that will be made precise below, the equation of motion (1.1), the boundary condition (1.2) and the equal time canonical commutation relations

$$[\Phi(t, x), \Phi(t, y)] = [\Phi^*(t, x), \Phi^*(t, y)] = 0 \quad , \quad (1.3)$$

$$[\Phi(t, x), \Phi^*(t, y)] = \delta(x - y) \quad , \quad (1.4)$$

where Φ^* is the Hermitian conjugate of Φ ;

3. To show the existence of a vacuum state Ω in the above mentioned domain, which is cyclic with respect to the field Φ^* .

The analogous construction in the case of the whole real line has been carried out some years ago⁴⁻⁹ by means of the quantum inverse scattering transform. The basic algebraic tool of this approach is the Zamolodchikov-Faddeev¹⁰ (ZF) algebra \mathcal{A}_R - an appropriate generalization of the canonical commutation relations which incorporates the two-body scattering matrix R . We will show below that the half line system can be treated in the framework of inverse scattering as well, the relevant algebraic structure being now the so called boundary algebra \mathcal{B}_R . In the same way as the ZF algebra has been conceived¹⁰ to represent the factorized scattering of integrable systems on the line, the general concept of boundary algebra¹¹ is inspired by Cherednik's scattering theory¹² of integrable systems on the half line. The fundamental feature of \mathcal{B}_R is that it encodes both the nontrivial scattering between particles and the reflection from the boundary at $x = 0$.

A preliminary account without proofs, which partially covers the results presented below, is given in Ref. 13. This paper is organized as follows. In the next section we summarize some known, but useful facts, about the classical NLS model both on \mathbf{R} and \mathbf{R}_+ . Sec. III represents a summary of those fundamental properties of \mathcal{B}_R and its Fock representations, which are needed in the quantization. In Sec. IV we define the quantum field $\Phi(t, x)$ and establish its kinematic properties, verifying the canonical commutation relations (1.3-4). The dynamics is investigated in Sec. V, where it is shown that Eqs.(1.1,2) are indeed satisfied. We sketch there also the derivation of the correlation functions. Sec. VI is devoted to the asymptotic theory of the NLS model on \mathbf{R}_+ . The last section contains our conclusions.

II. THE CLASSICAL NLS MODEL

The study of the classical NLS equation has a long story. Without entering the details, we will collect in this section some basic facts providing useful hints for the

quantization.

A. NLS on the real line

The equation of motion (1.1) on \mathbf{R} is obtained by varying the action

$$A[\Phi, \bar{\Phi}] = \int_{\mathbf{R}} dt \int_{\mathbf{R}} dx [i\bar{\Phi}(t, x)\partial_t \Phi(t, x) - |\partial_x \Phi(t, x)|^2 - g|\Phi(t, x)|^4] \quad . \quad (2.1)$$

The system admits an infinite number of integrals of motion, the energy

$$E[\Phi, \bar{\Phi}] = \int_{\mathbf{R}} dx [|\partial_x \Phi(t, x)|^2 + g|\Phi(t, x)|^4] \quad (2.2)$$

being one of them. Notice that $E[\Phi, \bar{\Phi}]$ is non-negative as long as $g \geq 0$. This constraint has an important role in the quantum version of the theory.

About twenty years ago Rosales¹⁴ discovered that Eq.(1.2) on \mathbf{R} admits solutions of the form

$$\Phi(t, x) = \sum_{n=0}^{\infty} (-g)^n \Phi^{(n)}(t, x) \quad , \quad (2.3)$$

where

$$\Phi^{(0)}(t, x) \equiv \tilde{\lambda}(t, x) = \int_{\mathbf{R}} \frac{dq}{2\pi} \lambda(q) e^{ixq - itq^2} \quad , \quad (2.4)$$

solves the free Schrödinger equation and

$$\Phi^{(n)}(t, x) = \int_{\mathbf{R}^{2n+1}} \prod_{\substack{i=1 \\ j=0}}^n \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} \bar{\lambda}(p_1) \cdots \bar{\lambda}(p_n) \lambda(q_n) \cdots \lambda(q_0) \frac{e^{i \sum_{j=0}^n (xq_j - tq_j^2) - i \sum_{i=1}^n (xp_i - tp_i^2)}}{\prod_{i=1}^n [(p_i - q_{i-1})(p_i - q_i)]} \quad . \quad (2.5)$$

The integration in (2.5) is defined by the principal value prescription and one assumes that $\lambda(k)$ is a function for which the integrals (2.4,5) exist and the series (2.3) converges uniformly in x for sufficiently small g . It is not difficult to argue that there is a large set of such functions; any λ belonging to the Schwartz test function space $\mathcal{S}(\mathbf{R})$ meets for instance the above requirements. In fact, expressing $\Phi^{(n)}(t, x)$ in terms of $\tilde{\lambda}(t, x)$, one finds

$$\Phi^{(n)}(t, x) = \int_{\mathbf{R}^{2n}} \left[\prod_{i=1}^n dy_i dz_i \tilde{\lambda}(t, y_i) \tilde{\lambda}(t, z_i) \right] \tilde{\lambda}(t, x + \sum_{i=1}^n y_i - z_i) \sigma(x; y_1, z_1, \dots, y_n, z_n) \quad , \quad (2.6)$$

where

$$\sigma(x; y_1, z_1, \dots, y_n, z_n) = 4^{-n} \prod_{i=1}^n \varepsilon \left(x + \sum_{j=1}^{i-1} y_j - \sum_{k=1}^i z_k \right) \varepsilon \left(\sum_{j=1}^i (y_j - z_j) \right), \quad (2.7)$$

and $\varepsilon(x)$ denotes the sign of x . Therefore,

$$|\Phi^{(n)}(t, x)| \leq \int_{\mathbf{R}^{2n}} \left[\prod_{i=1}^n dy_i dz_i |\tilde{\lambda}(t, y_i) \tilde{\lambda}(t, z_i)| \right] |\tilde{\lambda}(t, x + \sum_{i=1}^n y_i - z_i)|. \quad (2.8)$$

At the other hand, using standard estimates one can deduce that for any $\lambda(k) \in \mathcal{S}(\mathbf{R})$ there exist two positive constants Λ_1 and Λ_2 such that

$$\int_{\mathbf{R}} dx |\tilde{\lambda}(t, x)| \leq \Lambda_1 (1 + |t|) \quad , \quad \sup_{x \in \mathbf{R}} |\tilde{\lambda}(t, x)| \leq \Lambda_2 \quad . \quad (2.9)$$

Combining Eqs.(2.8) and (2.9) we conclude that the series (2.3) converges uniformly in x for

$$g < [\Lambda_1 (1 + |t|)]^{-2} \quad . \quad (2.10)$$

The main reason for focusing on the result of Rosales is because it turns out⁵⁻⁹ that the general structure of the solution (2.3-5) is preserved by the quantization. From this point of view it is instructive to investigate the behavior of (2.3-5) when the system is restricted on \mathbf{R}_+ .

B. NLS on the half line

The relative action, giving rise both to the equation of motion (1.1) on \mathbf{R}_+ and the boundary condition (1.2) is

$$A[\Phi, \bar{\Phi}] = \int_{\mathbf{R}} dt \int_{\mathbf{R}_+} dx [i\bar{\Phi}(t, x) \partial_t \Phi(t, x) - |\partial_x \Phi(t, x)|^2 - g|\Phi(t, x)|^4] - \eta \int_{\mathbf{R}} dt |\Phi(t, 0)|^2 \quad . \quad (2.11)$$

This action is invariant under time translations, which leads to conservation of the energy

$$E[\Phi, \bar{\Phi}] = \int_{\mathbf{R}_+} dx [|\partial_x \Phi(t, x)|^2 + g|\Phi(t, x)|^4] + \eta |\Phi(t, 0)|^2. \quad (2.12)$$

Positivity implies $g \geq 0$ and $\eta \geq 0$, which is the case we are going to analyze below.

The series (2.3), being a solution of the NLS equation on \mathbf{R} , is *a fortiori* a solution when restricted on \mathbf{R}_+ . In general however, it does not satisfy the boundary condition (1.2). In this respect, one has the following

Proposition 1: $\Phi(t, x)$ obeys the boundary condition (1.2), provided that $\lambda(k)$ satisfies

$$\lambda(k) = B(k)\lambda(-k) \quad , \quad (2.13)$$

where

$$B(k) = \frac{k - i\eta}{k + i\eta} \quad . \quad (2.14)$$

Proof. Using (2.13), we will show that $\Phi^{(n)}(t, x)$ satisfies (1.2) for any $n \geq 0$. For $n = 0$ the statement is obvious. So, let us focus on $\Phi^{(n)}(t, x)$ with $n \geq 1$. Changing variables in Eq.(2.5) according to

$$k_{2i-1} = p_i \quad , \quad k_{2j} = -q_j \quad , \quad i = 1, \dots, n; \quad j = 0, \dots, n, \quad (2.15)$$

one finds

$$\begin{aligned} & \lim_{x \downarrow 0} (\partial_x - \eta) \Phi^{(n)}(t, x) = \\ & \int_{\mathbf{R}^{2n+1}} \prod_{j=0}^{2n} \frac{dk_j}{2\pi} f^{(n)}(k_0, \dots, k_{2n}) \bar{\lambda}(k_1) \cdots \bar{\lambda}(k_{2n-1}) \lambda(-k_{2n}) \cdots \lambda(-k_0) e^{-it \sum_{j=0}^n (-1)^j k_j^2} , \end{aligned} \quad (2.16)$$

where

$$f^{(n)}(k_0, \dots, k_{2n}) = \frac{\sum_{j=0}^{2n} k_j - i\eta}{i \prod_{j=1}^{2n} (k_j + k_{j-1})} \quad . \quad (2.17)$$

Using the simple relations

$$B(k)B(-k) = B(k)\bar{B}(k) = 1 \quad , \quad (2.18)$$

one concludes that $f^{(n)}$ in Eq.(2.16) can be equivalently replaced by its B -symmetrized counterpart

$$f_B^{(n)}(k_0, \dots, k_{2n}) = \sum_{\sigma_0, \dots, \sigma_{2n} \in \{-1, 1\}} \frac{1}{4^n} \left(\prod_{j=0}^{2n} \frac{k_j + i\sigma_j \eta}{k_j + i\eta} \right) \frac{\sum_{j=0}^{2n} \sigma_j k_j - i\eta}{i \prod_{j=1}^{2n} (\sigma_j k_j + \sigma_{j-1} k_{j-1})} \quad . \quad (2.19)$$

We shall show now that $f_B^{(n)}$ vanishes identically. Eq.(2.19) can be given the more convenient form

$$f_B^{(n)}(k_0, \dots, k_{2n}) = \frac{N^{(n)}(k_0, \dots, k_{2n})}{4^n i \prod_{j=0}^{2n} (k_j + i\eta) \prod_{j=1}^{2n} (k_j^2 - k_{j-1}^2)}$$

where

$$N^{(n)}(k_0, \dots, k_{2n}) = \sum_{\sigma_0, \dots, \sigma_{2n} \in \{-1, 1\}} \prod_{j=0}^{2n} (k_j + i\sigma_j \eta) \prod_{j=1}^{2n} (\sigma_j k_j - \sigma_{j-1} k_{j-1}) \left(\sum_{j=0}^{2n} \sigma_j k_j - i\eta \right). \quad (2.20)$$

The final step is to prove then that the numerator $N^{(n)}$ vanishes. One way to show the validity of this quite remarkable identity, is to introduce the auxiliary function

$$M^{(n)}(k_0, \dots, k_{2n}) = \sum_{\sigma_0, \dots, \sigma_{2n} \in \{-1, 1\}} (\sigma_0 k_0 - i\eta) \prod_{j=0}^{2n} (k_j + i\sigma_j \eta) \prod_{j=1}^{2n} (\sigma_j k_j - \sigma_{j-1} k_{j-1}) \quad . \quad (2.21)$$

Now, after some algebra one derives the recurrence relations

$$N^{(n)}(k_0, \dots, k_{2n}) = -4k_0 k_1 (k_1^2 + \eta^2) N^{(n-1)}(k_2, \dots, k_{2n}) + 4k_0 k_1 (k_1^2 - k_0^2) M^{(n-1)}(k_2, \dots, k_{2n}) \quad , \quad (2.22)$$

$$M^{(n)}(k_0, \dots, k_{2n}) = -4k_0 k_1 (k_0^2 + \eta^2) M^{(n-1)}(k_2, \dots, k_{2n}) \quad . \quad (2.23)$$

Since $N^{(0)}(k_0) = M^{(0)}(k_0) = 0$, Eqs.(2.22,23) imply by induction that

$$N^{(n)}(k_0, \dots, k_{2n}) = 0 \quad , \quad M^{(n)}(k_0, \dots, k_{2n}) = 0 \quad , \quad (2.24)$$

which completes the argument.

We conclude here the brief introduction to the classical boundary value problem (1.1,2). Our next step will be to establish the quantum counterparts of the solution (2.3-5) and the constraint (2.13).

III. THE BOUNDARY ALGEBRA

As already mentioned in the introduction, our basic algebraic tool will be a particular associative algebra \mathcal{B}_R , whose generators satisfy specific quadratic relations.

A. Definition of \mathcal{B}_R

The concept of boundary algebra has been introduced and investigated in a general context in Ref. 11. Here we will consider the following special case. Let $R : \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{C}$ be a measurable function satisfying

$$R(k_1, k_2)R(k_2, k_1) = R(k_1, k_2)\overline{R}(k_1, k_2) = 1 \quad . \quad (3.1)$$

The boundary algebra \mathcal{B}_R is generated by the operator valued distributions $\{a(k), a^*(k), b(k) : k \in \mathbf{R}\}$, satisfying quadratic exchange relations, which can be conveniently grouped in two sets. The first one is

$$a(k_1)a(k_2) - R(k_2, k_1)a(k_2)a(k_1) = 0 \quad , \quad (3.2)$$

$$a^*(k_1)a^*(k_2) - R(k_2, k_1)a^*(k_2)a^*(k_1) = 0 \quad , \quad (3.3)$$

$$a(k_1)a^*(k_2) - R(k_1, k_2)a^*(k_2)a(k_1) = 2\pi\delta(k_1 - k_2) + b(k_1)2\pi\delta(k_1 + k_2) \quad . \quad (3.4)$$

The second set of constraints describes the exchange relations of $b(k)$ and reads

$$a(k_1)b(k_2) = R(k_2, k_1)R(k_1, -k_2)b(k_2)a(k_1) \quad , \quad (3.5)$$

$$b(k_2)a^*(k_1) = R(k_2, k_1)R(k_1, -k_2)a^*(k_1)b(k_2) \quad , \quad (3.6)$$

$$b(k_1)b(k_2) = b(k_2)b(k_1) \quad . \quad (3.7)$$

Notice that if we formally set $b(k) \rightarrow 0$, the relations (3.5-7) trivialize, while (3.2-4) reproduce the defining relations of the ZF algebra \mathcal{A}_R . As it is well known, the factorized scattering of 1+1 dimensional integrable systems is encoded in \mathcal{A}_R , i.e. in a boundary algebra in which the so called boundary operator $b(k)$ is trivially implemented. On the contrary, it turns out¹¹ that whenever there is a reflecting boundary, one needs a *reflection* boundary algebra, i.e. a boundary algebra with the additional constraint

$$b(k)b(-k) = 1 \quad , \quad (3.8)$$

which obviously prevents the boundary operator from being zero. In the case of the NLS on the half line, we shall need a reflection boundary algebra \mathcal{B}_R with exchange factor

$$R(k_1, k_2) = \frac{k_1 - k_2 - ig}{k_1 - k_2 + ig} \quad , \quad (3.9)$$

where $g \geq 0$ is the coupling constant of the NLS model. $R(k_1, k_2)$ is actually the two-body bulk scattering matrix of the NLS model^{4–9} and satisfies (3.1).

B. Fock Representations

Following some basic ideas of Ref. 15, we have constructed in Ref. 11 the Fock representations of \mathcal{B}_R . These representations are characterized by the existence of a vacuum state Ω , which is cyclic with respect $a^*(k)$ and satisfies

$$a(k)\Omega = 0 \quad . \quad (3.10)$$

In the reflection case (3.8), the vacuum is¹¹ always an eigenvector of the boundary operator $b(k)$, i.e.

$$b(k)\Omega = B(k)\Omega \quad , \quad (3.11)$$

where $B(k)$ is a measurable function obeying Eq.(2.18). Conversely, any $B(k)$ of this type defines a Fock representation on a Hilbert space $\mathcal{F}_{R,B}$, whose vacuum satisfies (3.11). We will show below that the state space $\mathcal{H}_{g,\eta}$ of the NLS model on \mathbf{R}_+ is

$$\mathcal{H}_{g,\eta} = \mathcal{F}_{R,B} \quad , \quad (3.12)$$

with B and R given by (2.14) and (3.9) respectively. The mere fact that our system has a boundary shows up at the algebraic level, turning the ZF algebra into a reflection boundary algebra \mathcal{B}_R , i.e. forcing a non zero boundary operator $b(k)$. The details of the boundary condition (the value of the parameter η) enter at the representation level through the reflection coefficient $B(k)$. In the Fock space $\mathcal{F}_{R,B}$ one has

$$a(k) = b(k)a(-k) \quad , \quad (3.13)$$

$$a^*(k) = a^*(-k)b(-k) \quad , \quad (3.14)$$

which descend from a peculiar automorphism of \mathcal{B}_R , established in Ref. 11. The relation (3.13) turns out to be the correct quantum analogue of Eq.(2.13). Let us stress once more that the c-number reflection coefficient $B(k)$ must be distinguished from the boundary generator $b(k)$, which according to Eqs.(3.5,6) does not even commute with $\{a(k), a^*(k)\}$.

To the end of this section we will give some details about the structure of $\mathcal{F}_{R,B}$ which are needed for our construction. One has

$$\mathcal{F}_{R,B} \equiv \bigoplus_{n=0}^{\infty} \mathcal{H}_{R,B}^n \quad , \quad (3.15)$$

where $\mathcal{H}_{R,B}^0 \equiv \mathbf{C}$ and the n -particle space $\mathcal{H}_{R,B}^n$ with $n \geq 1$ is a subspace of $L^2(\mathbf{R}^n)$ defined as follows:

(i) a L^2 -function $\varphi(p_1)$ belongs to $\mathcal{H}_{R,B}^1$ if and only if

$$\varphi(p_1) = B(p_1)\varphi(-p_1) \quad ; \quad (3.16)$$

(ii) a L^2 -function $\varphi(p_1, \dots, p_n)$ with $n \geq 2$ belongs to $\mathcal{H}_{R,B}^n$ if and only if

$$\varphi(p_1, \dots, p_{n-1}, p_n) = B(p_n)\varphi(p_1, \dots, p_{n-1}, -p_n) \quad , \quad (3.17)$$

and

$$\varphi(p_1, \dots, p_i, p_{i+1}, \dots, p_n) = R(p_i, p_{i+1})\varphi(p_1, \dots, p_{i+1}, p_i, \dots, p_n) \quad , \quad (3.18)$$

for any $1 \leq i \leq n-1$.

Eqs.(3.16-18) define a closed subspace $\mathcal{H}_{R,B}^n \subset L^2(\mathbf{R}^n)$. We will denote by $P_{R,B}^{(n)}$ the corresponding orthogonal projection operator. We introduce also the finite particle space $\mathcal{F}_{R,B}^0 \subset \mathcal{F}_{R,B}$, generated by $\{\mathcal{H}_{R,B}^n : n = 0, 1, \dots\}$. We recall that $\mathcal{F}_{R,B}^0$ is the linear space of sequences $\varphi = (\varphi^{(0)}, \varphi^{(1)}, \dots, \varphi^{(n)}, \dots)$ with $\varphi^{(n)} \in \mathcal{H}_{R,B}^n$ and $\varphi^{(n)} = 0$ for n large enough. The vacuum state is $\Omega = (1, 0, \dots, 0, \dots)$. The L^2 -scalar product on $\mathcal{H}_{R,B}^n$ defines in the standard way the scalar product $\langle \cdot, \cdot \rangle$ in the (Hilbert) direct sum (3.15).

At this point we are in position to define on $\mathcal{F}_{R,B}^0$ the annihilation and creation operators $\{a(f), a^*(f) : f \in L^2(\mathbf{R})\}$. We set $a(f)\Omega = 0$ and

$$[a(f)\varphi]^{(n)}(p_1, \dots, p_n) = \sqrt{n+1} \int_{\mathbf{R}} \frac{dp}{2\pi} \bar{f}(p) \varphi^{(n+1)}(p, p_1, \dots, p_n) \quad , \quad (3.19)$$

$$[a^*(f)\varphi]^{(n)}(p_1, \dots, p_n) = \sqrt{n} \left[P_{R,B}^{(n)} f \otimes \varphi^{(n-1)} \right](p_1, \dots, p_n) \quad , \quad (3.20)$$

for all $\varphi \in \mathcal{F}_{R,B}^0$. The operators $a(f)$ and $a^*(f)$ are in general unbounded on $\mathcal{F}_{R,B}^0$. One can easily see however that $a(f)$ and $a^*(f)$ are bounded on each $\mathcal{H}_{R,B}^n$. In fact, for all $\varphi \in \mathcal{H}_{R,B}^n$ one has the estimates

$$\|a(f)\varphi\| \leq \sqrt{n}\|f\|\|\varphi\|, \quad \|a^*(f)\varphi\| \leq \sqrt{n+1}\|f\|\|\varphi\|, \quad (3.21)$$

$\|\cdot\|$ being the L^2 -norm. Notice also that $a^*(f)$ is linear in f , whereas $a(f)$ is antilinear. The operator-valued distributions $a(p)$ and $a^*(p)$, generating the Fock representation of \mathcal{B}_R , are defined by

$$a(f) = \int_{\mathbf{R}} \frac{dp}{2\pi} \bar{f}(p) a(p) \quad , \quad a^*(f) = \int_{\mathbf{R}} \frac{dp}{2\pi} f(p) a^*(p) \quad , \quad (3.22)$$

and are related by Hermitian conjugation, namely

$$\langle \varphi, a(f)\psi \rangle = \langle a^*(f)\varphi, \psi \rangle, \quad \forall \varphi, \psi \in \mathcal{F}_{R,B}^0. \quad (3.23)$$

Finally, the action of the boundary generator $b(p)$ on $\mathcal{F}_{R,B}^0$ is defined by Eq.(3.11) and

$$\begin{aligned} [b(p)\varphi]^{(n)}(p_1, \dots, p_n) = \\ [R(p, p_1)R(p, p_2) \cdots R(p, p_n)B(p)R(p_n, -p) \cdots R(p_2, -p)R(p_1, -p)]\varphi^{(n)}(p_1, \dots, p_n) \quad . \end{aligned} \quad (3.24)$$

One can show¹¹ that $\{a(p), a^*(p), b(p)\}$, defined above, indeed satisfy the exchange relations (3.2-7) and the reflection condition (3.8). Moreover, the vacuum Ω obeys the requirements formulated in the beginning of this subsection.

It is convenient to introduce here a domain $\mathcal{D} \subset \mathcal{F}_{R,B}$, which will be frequently used in what follows. Setting

$$\mathcal{D}^0 \equiv \mathbf{C}, \quad \mathcal{D}^n \equiv \left\{ \int_{\mathbf{R}^n} dp_1 \dots dp_n f(p_1, \dots, p_n) a^*(p_1) \dots a^*(p_n) \Omega : f \in \mathcal{S}(\mathbf{R}^n), n \geq 1 \right\}, \quad (3.25)$$

we define \mathcal{D} to be the linear space of sequences $\varphi = (\varphi^{(0)}, \varphi^{(1)}, \dots, \varphi^{(n)}, \dots)$, where $\varphi^{(n)} \in \mathcal{D}^n$ and $\varphi^{(n)}$ vanish for n large enough. By construction \mathcal{D} is a proper subspace of $\mathcal{F}_{R,B}^0$. Nevertheless, \mathcal{D} is dense in $\mathcal{F}_{R,B}$ as well. Indeed, using that the factors R and B are smooth (i.e. C^∞) bounded functions, one has that \mathcal{D}^n is dense in $\mathcal{H}_{R,B}^n$, which implies the statement. We observe that

$$a(f)\mathcal{D} \subset \mathcal{D}, \quad a^*(f)\mathcal{D} \subset \mathcal{D}, \quad \forall f \in \mathcal{S}(\mathbf{R}) \quad . \quad (3.26)$$

Notice also that the matrix elements of $a^*(k)$ between states from \mathcal{D} are smooth functions of k . More generally, one has

$$\langle \varphi, a^*(k_1) \cdots a^*(k_n) \psi \rangle \in \mathcal{S}(\mathbf{R}^n), \quad \forall \varphi, \psi \in \mathcal{D}. \quad (3.27)$$

Summarizing, we introduced in this section the boundary algebra \mathcal{B}_R and its Fock representation $\mathcal{F}_{R,B}$, which are the main ingredients in the construction of the quantum solution of the boundary value problem (1.1,2).

IV. QUANTIZATION

A. The quantum field $\Phi(t, x)$

Our first step will be to introduce the quantum analog of $\Phi^{(n)}(t, x)$. For this purpose we consider

$$\Phi^{(0)}(t, x) \equiv \tilde{a}(t, x) = \int_{\mathbf{R}} \frac{dq}{2\pi} a(q) e^{ixq - itq^2}, \quad (4.1)$$

$$\begin{aligned} \Phi^{(n)}(t, x) = \\ \int_{\mathbf{R}^{2n+1}} \prod_{\substack{i=1 \\ j=0}}^n \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} a^*(p_1) \cdots a^*(p_n) a(q_n) \cdots a(q_0) \frac{e^{i \sum_{j=0}^n (xq_j - tq_j^2) - i \sum_{i=1}^n (xp_i - tp_i^2)}}{\prod_{i=1}^n [(p_i - q_{i-1} - i\epsilon)(p_i - q_i - i\epsilon)]}, \end{aligned} \quad (4.2)$$

thus replacing formally $\{\lambda(p), \bar{\lambda}(p)\}$ in Eqs.(2.4,5) by the generators $\{a(p), a^*(p)\}$ of \mathcal{B}_R in the Fock representation $\mathcal{F}_{R,B}$ and fixing an $i\epsilon$ prescription to contour poles. Our first task will be to give meaning of $\Phi^{(n)}(t, x)$ as a quadratic form in \mathcal{D} :

Proposition 2: For any $\varphi, \psi \in \mathcal{D}$, the expectation value

$$\langle \varphi, \Phi^{(n)}(t, x) \psi \rangle, \quad (4.3)$$

is a C^∞ function of t, x .

Proof. The case $n = 0$ is trivial. For $n \geq 1$ it is enough to take $\varphi \in \mathcal{D}^m$ and $\psi \in \mathcal{D}^{m+1}$ with $m > n$. Some elementary algebra leads to

$$\langle \varphi, \Phi^{(n)}(t, x) \psi \rangle = \int_{\mathbf{R}^{m+n+1}} \prod_{i_1=1}^n \frac{dp_{i_1}}{2\pi} \prod_{i_2=0}^n \frac{dq_{i_2}}{2\pi} \prod_{i_3=n+1}^m \frac{dk_{i_3}}{2\pi}$$

$$\overline{\varphi}(p_1, \dots, p_n, k_{n+1}, \dots, k_m) \frac{e^{i \sum_{j=0}^n (xq_j - tq_j^2) - i \sum_{i=1}^n (xp_i - tp_i^2)}}{\prod_{i=1}^n [(p_i - q_{i-1} - i\epsilon)(p_i - q_i - i\epsilon)]} \psi(q_0, \dots, q_n, k_{n+1}, \dots, k_m), \quad (4.4)$$

which, using that φ and ψ are Schwartz test functions, implies the proposition.

Taking into account that \mathcal{D} contains only finite particle vectors, we conclude that also $\Phi(t, x)$ is a quadratic form on \mathcal{D} , smooth in both t and x . The conjugate $\Phi^*(t, x)$ is defined by

$$\langle \varphi, \Phi^*(t, x) \psi \rangle = \overline{\langle \psi, \Phi(t, x) \varphi \rangle}, \quad (4.5)$$

which is of course smooth in t and x as well. The counterparts of Eqs.(4.1,2) read

$$\Phi^{*(0)}(t, x) \equiv \tilde{a}^*(t, x) = \int_{\mathbf{R}} \frac{dq}{2\pi} a^*(q) e^{-ixq + itq^2}, \quad (4.6)$$

$$\begin{aligned} \Phi^{*(n)}(t, x) = \\ \int_{\mathbf{R}^{2n+1}} \prod_{\substack{i=1 \\ j=0}}^n \frac{dp_i}{2\pi} \frac{dq_j}{2\pi} a^*(q_0) \cdots a^*(q_n) a(p_n) \cdots a(p_1) \frac{e^{i \sum_{i=1}^n (xp_i - tp_i^2) - i \sum_{j=0}^n (xq_j - tq_j^2)}}{\prod_{i=1}^n [(p_i - q_{i-1} + i\epsilon)(p_i - q_i + i\epsilon)]}. \end{aligned} \quad (4.7)$$

Since the system we are considering is in \mathbf{R}_+ , we adopt the smearing

$$\Phi(t, f) = \int dx \overline{f}(x) \Phi(t, x), \quad \Phi^*(t, f) = \int dx f(x) \Phi^*(t, x), \quad f \in C_0^\infty(\mathbf{R}_+) \quad , \quad (4.8)$$

where $C_0^\infty(\mathbf{R}_+)$ is the set of infinitely differentiable functions with compact support in \mathbf{R}_+ . Again, $\Phi(t, f)$ and $\Phi^*(t, f)$ have meaning as quadratic forms on \mathcal{D} , which are related by

$$\langle \varphi, \Phi^*(t, f) \psi \rangle = \overline{\langle \psi, \Phi(t, f) \varphi \rangle}. \quad (4.9)$$

In order to formulate some other less obvious properties of $\Phi(t, f)$ and $\Phi^*(t, f)$, we have to introduce the following partial ordering relation in $C_0^\infty(\mathbf{R}_+)$. Let $f_1, f_2 \in C_0^\infty(\mathbf{R}_+)$. Then

$$f_1 \prec f_2 \iff x_1 < x_2 \quad \forall x_1 \in \text{supp} f_1, \quad \forall x_2 \in \text{supp} f_2 \quad . \quad (4.10)$$

Instead of $f_1 \prec f_2$, we will also write $f_2 \succ f_1$. Denoting by $\tilde{a}^*(t, f)$ the operator

$$\tilde{a}^*(t, f) = \int dx f(x) \tilde{a}^*(t, x) \quad , \quad (4.11)$$

one can prove the following technical

Lemma 1: Let $\varphi, \psi \in \mathcal{D}$.

(a) The identity

$$\langle \varphi, \Phi^*(t, h) \tilde{a}^*(t, f) \psi \rangle = \langle \varphi, \tilde{a}^*(t, f) \Phi^*(t, h) \psi \rangle \quad , \quad (4.12)$$

holds if $h \prec f$;

(b) One has

$$\langle \varphi, \Phi^*(t, h) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega \rangle = \langle \varphi, \tilde{a}^*(t, h) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega \rangle \quad , \quad (4.13)$$

provided that $h \succ f_j$ for any $j = 1, \dots, n$;

(c) For any $f_1 \succ f_2 \succ \dots \succ f_n$, one has

$$\begin{aligned} & \langle \varphi, \Phi(t, h) \tilde{a}^*(t, f_1) \tilde{a}^*(t, f_2) \cdots \tilde{a}^*(t, f_n) \Omega \rangle = \\ & \sum_{j=1}^n (h, f_j) \langle \varphi, \tilde{a}^*(t, f_1) \cdots \widehat{\tilde{a}^*(t, f_j)} \cdots \tilde{a}^*(t, f_n) \Omega \rangle , \end{aligned} \quad (4.14)$$

where (\cdot, \cdot) denotes the L^2 -scalar product and the hat indicates that the corresponding field must be omitted.

Proof. The proof of the identities (4.12-14) is analogous to that given by Davies⁸ for the NLS on \mathbf{R} , so we skip it. We only remark that the novelty on \mathbf{R}_+ consists in evaluating the contributions of the boundary generator b , which stem from the exchange of a and a^* . It is easy to see that these contributions actually vanish, due to the support requirements imposed on the test functions and the condition $\eta \geq 0$.

Summarizing, $\Phi(t, f)$ and $\Phi^*(t, f)$ have been so far defined as quadratic forms on \mathcal{D} and are Schwartz distributions with respect to f . Our main goal to the end of this subsection will be to show that $\Phi(t, f)$ and $\Phi^*(t, f)$ are actually well defined operators. In order to construct a common invariant domain for these operators, we introduce the subspace

$$\mathcal{D}_0^n \equiv \text{sp} \{ \tilde{a}^*(t, f_1) \tilde{a}^*(t, f_2) \cdots \tilde{a}^*(t, f_n) \Omega : f_1 \succ f_2 \succ \cdots \succ f_n \} \subset \mathcal{H}_{R,B}^n, \quad n \geq 1, \quad (4.15)$$

where sp indicates the linear span and $t \in \mathbf{R}$ is arbitrary but fixed. Setting $\mathcal{D}_0^0 = \mathbf{C}$, we define \mathcal{D}_0 to be the linear space of sequences $\varphi = (\varphi^{(0)}, \varphi^{(1)}, \dots, \varphi^{(n)}, \dots)$ with $\varphi^{(n)} \in \mathcal{D}_0^n$ and $\varphi^{(n)} = 0$ for n large enough. Both \mathcal{D} and \mathcal{D}_0 are subspaces of the finite particle space $\mathcal{F}_{R,B}^0$. We know already that \mathcal{D} is dense in $\mathcal{F}_{R,B}$. Although it is less obvious, the same is true for \mathcal{D}_0 .

Proposition 3: \mathcal{D}_0 is dense in $\mathcal{F}_{R,B}$.

Proof. It is enough to demonstrate that the space \mathcal{D}_0^n is dense in $\mathcal{H}_{R,B}^n$ for any $t \in \mathbf{R}$ and $n \geq 1$. So, let us consider the matrix element

$$\tilde{A}_{t,\varphi}(x_1, \dots, x_n) \equiv \langle \varphi, \tilde{a}^*(t, x_1) \cdots \tilde{a}^*(t, x_n) \Omega \rangle \quad , \quad (4.16)$$

where $\varphi \in \mathcal{D}^n$ is arbitrary. According to Eq.(3.27) $\tilde{A}_{t,\varphi} \in \mathcal{S}(\mathbf{R}^n)$. In order to prove the statement, it is sufficient to show that

$$\tilde{A}_{t,\varphi}(x_1, \dots, x_n) = 0, \quad \forall x_1 > x_2 > \dots > x_n > 0, \quad (4.17)$$

implies $\varphi = 0$. It is convenient for this purpose to investigate

$$A_{t,\varphi}(p_1, \dots, p_n) \equiv \int_{\mathbf{R}^n} \prod_{j=1}^n dx_j e^{i \sum_{j=1}^n p_j x_j} \tilde{A}_{t,\varphi}(x_1, \dots, x_n) = e^{it \sum_{j=1}^n p_j^2} \langle \varphi, a^*(p_1) \cdots a^*(p_n) \Omega \rangle \in \mathcal{S}(\mathbf{R}^n). \quad (4.18)$$

The behavior of this function under the reflection of one of its arguments or the exchange of two consecutive arguments is determined by Eqs.(3.3,6,11,14). Using this fact, one can verify that the function

$$B_{t,\varphi}(p_1, \dots, p_n) \equiv \Lambda(p_1, \dots, p_n) A_{t,\varphi}(p_1, \dots, p_n) \quad , \quad (4.19)$$

where

$$\Lambda(p_1, \dots, p_n) \equiv \prod_{j=1}^n \left[(p_j - i\eta) \prod_{\substack{k=1 \\ k > j}}^n (p_j - p_k - ig)(p_j + p_k - ig) \right] \quad (4.20)$$

satisfies

$$B_{t,\varphi}(p_1, \dots, p_j, \dots, p_n) = -B_{t,\varphi}(p_1, \dots, -p_j, \dots, p_n), \quad \forall j = 1, \dots, n, \quad (4.21)$$

$$B_{t,\varphi}(p_1, \dots, p_j, p_{j+1}, \dots, p_n) = -B_{t,\varphi}(p_1, \dots, p_{j+1}, p_j, \dots, p_n), \quad \forall j = 1, \dots, n-1. \quad (4.22)$$

By construction $B_{t,\varphi} \in \mathcal{S}(\mathbf{R}^n)$ and

$$\begin{aligned} \tilde{B}_{t,\varphi}(x_1, \dots, x_n) = \\ \int_{\mathbf{R}^n} \prod_{j=1}^n \frac{dp_j}{2\pi} e^{-i \sum_{j=1}^n p_j x_j} B_{t,\varphi}(p_1, \dots, p_n) = \Lambda(i\partial_1, \dots, i\partial_n) \tilde{A}_{t,\varphi}(x_1, \dots, x_n) \quad , \end{aligned} \quad (4.23)$$

admits the same antisymmetry properties as $B_{t,\varphi}$. Therefore, using the smoothness of $\tilde{A}_{t,\varphi}$ and Eq.(4.17), we deduce that $\tilde{B}_{t,\varphi}$ vanishes identically, or equivalently,

$$B_{t,\varphi}(p_1, \dots, p_n) = 0, \quad \forall p_j \in \mathbf{R}. \quad (4.24)$$

Combining Eqs.(4.18,19,24) with the fact that $\Lambda(p_1, \dots, p_n) \neq 0$ for any $p_j \in \mathbf{R}$, one gets

$$\langle \varphi, a^*(p_1) \cdots a^*(p_n) \Omega \rangle = 0, \quad \forall p_j \in \mathbf{R}, \quad (4.25)$$

which, because of the cyclicity of Ω with respect to a^* , implies $\varphi = 0$. This concludes the argument.

It is convenient in what follows to have an explicit formula for the scalar product in \mathcal{D}_0 .

It is provided by the following

Lemma 2: Let $f_1 \succ f_2 \succ \cdots \succ f_n$ and $h_1 \succ h_2 \succ \cdots \succ h_n$. Then

$$\langle \tilde{a}^*(t, h_1) \cdots \tilde{a}^*(t, h_n) \Omega, \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega \rangle = (h_1 \otimes \cdots \otimes h_n, f_1 \otimes \cdots \otimes f_n) \quad . \quad (4.26)$$

Proof. It is enough to expand the left hand side, using the algebraic relations (3.4) and Eq.(3.10). Taking into account the support properties of the test functions involved, all terms, except the one in the right hand side of (4.26), vanish.

A simple corollary of the previous lemma is now in order. Since any $\varphi \in \mathcal{D}_0^n$ can be represented as

$$\varphi = \sum_{\alpha \in A} \tilde{a}^*(t, f_1^\alpha) \cdots \tilde{a}^*(t, f_n^\alpha) \Omega \quad ,$$

where A is a finite set and $f_1^\alpha \succ f_2^\alpha \succ \cdots \succ f_n^\alpha$ for all $\alpha \in A$, one has that

$$\langle \varphi, \varphi \rangle^2 \equiv \|\varphi\|^2 = \left\| \sum_{\alpha \in A} f_1^\alpha \otimes \cdots \otimes f_n^\alpha \right\|^2 \quad . \quad (4.27)$$

We are now in position to show the following

Proposition 4: The estimate

$$|\langle \varphi, \Phi(t, f)\psi \rangle| \leq (n+1)\|f\| \|\varphi\| \|\psi\| \quad (4.28)$$

holds for any $\varphi \in \mathcal{D}_0^n$, $\psi \in \mathcal{D}_0^{n+1}$ and $f \in C_0^\infty(\mathbf{R}_+)$.

Proof. Let

$$\varphi = \sum_{\alpha \in A} \tilde{a}^*(t, f_1^\alpha) \cdots \tilde{a}^*(t, f_n^\alpha) \Omega, \quad \psi = \sum_{\beta \in B} \tilde{a}^*(t, h_0^\beta) \cdots \tilde{a}^*(t, h_n^\beta) \Omega, \quad (4.29)$$

with $f_1^\alpha \succ f_2^\alpha \succ \cdots \succ f_n^\alpha$ and $h_0^\beta \succ h_2^\beta \succ \cdots \succ h_n^\beta$. Then

$$\begin{aligned} \langle \varphi, \Phi(t, f)\psi \rangle &= \sum_{\alpha \in A} \sum_{\beta \in B} \langle \tilde{a}^*(t, f_1^\alpha) \cdots \tilde{a}^*(t, f_n^\alpha) \Omega, \Phi(t, f) \tilde{a}^*(t, h_0^\beta) \cdots \tilde{a}^*(t, h_n^\beta) \Omega \rangle \\ &= \sum_{\alpha \in A} \sum_{\beta \in B} \sum_{j=0}^n (f, h_j^\alpha) (f_1^\alpha \otimes \cdots \otimes f_n^\alpha, h_0^\beta \otimes \cdots \otimes \hat{h}_j^\beta \otimes \cdots \otimes h_n^\beta) \\ &= \sum_{j=0}^n \left(\sum_{\alpha \in A} f_1^\alpha \otimes \cdots \otimes f_{j-1}^\alpha \otimes f \otimes f_j^\alpha \cdots \otimes f_n^\alpha, \sum_{\beta \in B} h_0^\beta \otimes \cdots \otimes h_j^\beta \otimes \cdots \otimes h_n^\beta \right), \end{aligned}$$

where use has been made of point (c) of Lemma 1. Applying now the Minkowski inequality, one finds

$$|\langle \varphi, \Phi(t, f)\psi \rangle| \leq \sum_{j=0}^n \|f\| \left\| \sum_{\alpha \in A} f_1^\alpha \otimes \cdots \otimes f_n^\alpha \right\| \left\| \sum_{\beta \in B} h_0^\beta \otimes \cdots \otimes h_n^\beta \right\| \leq (n+1)\|f\| \|\varphi\| \|\psi\|. \quad (4.30)$$

The above proposition shows that $\Phi(t, f)$, considered as quadratic form, is bounded on $\mathcal{D}_0^n \times \mathcal{D}_0^{n+1}$ and defines therefore a bounded operator $\mathcal{H}_{R,B}^{n+1} \rightarrow \mathcal{H}_{R,B}^n$. Since this occurs for any $n \geq 0$, we recover an operator $\Phi(t, f) : \mathcal{F}_{R,B}^0 \rightarrow \mathcal{F}_{R,B}^0$, whose properties are collected in

Theorem 1: $\Phi(t, f) : \mathcal{F}_{R,B}^0 \rightarrow \mathcal{F}_{R,B}^0$ is a linear operator, satisfying

$$\Phi(t, f)\Omega = 0, \quad \Phi(t, f) : \mathcal{H}_{R,B}^{n+1} \rightarrow \mathcal{H}_{R,B}^n, \quad n \geq 0. \quad (4.31)$$

Moreover, for any $\varphi, \psi \in \mathcal{F}_{R,B}^0$, the matrix element $\langle \varphi, \Phi(t, f)\psi \rangle$ has the following properties:

- i) It is antilinear and L^2 -continuous in f ;
- ii) It is continuous in $t \in \mathbf{R}$;
- iii) It is smooth in $t \in \mathbf{R}$, provided that $\varphi, \psi \in \mathcal{D}$.

Proof: All the statements are simple corollaries of the above propositions.

The operator $\Phi(t, f)$ is densely defined and admits therefore a Hermitian conjugate $\Phi^*(t, f)$.

Theorem 2 : *The field $\Phi^*(t, f)$ satisfies*

$$\Phi^*(t, f)\Omega = \tilde{a}^*(t, f)\Omega, \quad \Phi^*(t, f) : \mathcal{H}_{R,B}^n \rightarrow \mathcal{H}_{R,B}^{n+1}, \quad n \geq 0 \quad (4.32)$$

and therefore leaves $\mathcal{F}_{R,B}^0$ invariant. Moreover

$$\langle \varphi, \Phi(t, f)\psi \rangle = \langle \Phi^*(t, f)\varphi, \psi \rangle, \quad (4.33)$$

holds for any $\varphi, \psi \in \mathcal{F}_{R,B}^0$.

Proof: One uses the fact that $\Phi(t, f)$ is bounded on each $\mathcal{H}_{R,B}^n$.

We will show now that the operators $\Phi(t, f)$ and $\Phi^*(t, f)$ satisfy the basic requirements for nonrelativistic quantum fields.

B. Cyclicity of Ω and commutation relations

We start with

Theorem 3 (Cyclicity) : *The vacuum Ω is a cyclic vector for the field Φ^* . More precisely the space*

$$\mathcal{E}_0^n \equiv \text{sp} \{ \Phi^*(t, f_1)\Phi^*(t, f_2) \cdots \Phi^*(t, f_n)\Omega : f_1 \prec f_2 \prec \cdots \prec f_n \},$$

is dense in $\mathcal{H}_{R,B}^n$.

Proof. Using Eqs.(4.12-13) of Lemma 1, one easily proves by induction that

$$\Phi^*(t, f_1)\Phi^*(t, f_2) \cdots \Phi^*(t, f_n)\Omega = \tilde{a}^*(t, f_n) \cdots \tilde{a}^*(t, f_1)\Omega, \quad (4.34)$$

as long as $f_1 \prec f_2 \prec \dots \prec f_n$. Thus $\mathcal{E}_0^n = \mathcal{D}_0^n$, and the statement follows directly from Proposition 3.

Remark: Theorem 3 is slightly stronger than the standard cyclicity¹⁶, because of the ordering among the functions f_1, \dots, f_n required in the definition of \mathcal{E}_0^n .

Let us consider now the canonical commutation relations (1.3,4). We shall prove

Theorem 4 : *The equal time canonical commutation relations*

$$[\Phi(t, h_1), \Phi(t, h_2)] = [\Phi^*(t, h_1), \Phi^*(t, h_2)] = 0 \quad , \quad (4.35)$$

$$[\Phi(t, h_1), \Phi^*(t, h_2)] = (h_1, h_2) \quad , \quad (4.36)$$

hold on $\mathcal{F}_{R,B}^0$ for any $h_1, h_2 \in \mathcal{S}(\mathbf{R}_+)$.

Proof: In order to demonstrate Eq.(4.35), we observe that Eq.(4.14) implies

$$\Phi(t, h_2) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega = \sum_{j=1}^n (h_2, f_j) \tilde{a}^*(t, f_1) \cdots \widehat{\tilde{a}^*(t, f_j)} \cdots \tilde{a}^*(t, f_n) \Omega ,$$

where $f_1 \succ \dots \succ f_n$. Therefore

$$\begin{aligned} & \Phi(t, h_1) \Phi(t, h_2) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega = \\ & \sum_{j=1}^n \sum_{\substack{k=1 \\ k \neq j}}^n (h_2, f_j) (h_1, f_k) \tilde{a}^*(t, f_1) \cdots \widehat{\tilde{a}^*(t, f_j)} \cdots \widehat{\tilde{a}^*(t, f_k)} \cdots \tilde{a}^*(t, f_n) \Omega , \end{aligned} \quad (4.37)$$

which, being symmetric under the exchange of h_1 with h_2 , implies the vanishing of $[\Phi(t, h_1), \Phi(t, h_2)]$ on \mathcal{D}_0^n . Then one extends by continuity to $\mathcal{H}_{R,B}^n$ and by linearity to $\mathcal{F}_{R,B}^0$. The validity of $[\Phi^*(t, h_1), \Phi^*(t, h_2)] = 0$ follows by Hermitian conjugation.

We turn now to Eq.(4.36). Let $f_1 \succ \dots \succ f_n$ and $h_1, h_2 \in \mathcal{S}(\mathbf{R}_+)$. Assume that

$$f_k \succ h_2 \succ f_{k+1} \quad . \quad (4.38)$$

Using Lemma 1, one gets

$$\begin{aligned} & \Phi(t, h_1) \Phi^*(t, h_2) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega = \\ & \Phi(t, h_1) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_k) \Phi^*(t, h_2) \tilde{a}^*(t, f_{k+1}) \cdots \tilde{a}^*(t, f_n) \Omega = \end{aligned}$$

$$\begin{aligned}
& \Phi(t, h_1) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_k) \tilde{a}^*(t, h_2) \tilde{a}^*(t, f_{k+1}) \cdots \tilde{a}^*(t, f_n) \Omega = \\
& = (h_1, h_2) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega + \\
& \sum_{j=1}^n (h_1, f_j) \tilde{a}^*(t, f_1) \cdots \widehat{\tilde{a}^*(t, f_j)} \cdots \tilde{a}^*(t, f_k) \tilde{a}^*(t, h_2) \tilde{a}^*(t, f_{k+1}) \cdots \tilde{a}^*(t, f_n) \Omega \quad .
\end{aligned}$$

Analogously,

$$\begin{aligned}
& \Phi^*(t, h_2) \Phi(t, h_1) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega = \\
& \sum_{j=1}^n (h_1, f_j) \tilde{a}^*(t, f_1) \cdots \widehat{\tilde{a}^*(t, f_j)} \cdots \tilde{a}^*(t, f_k) \tilde{a}^*(t, h_2) \tilde{a}^*(t, f_{k+1}) \cdots \tilde{a}^*(t, f_n) \Omega \quad .
\end{aligned}$$

Therefore

$$[\Phi(t, h_1), \Phi^*(t, h_2)] \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega = (h_1, h_2) \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega \quad . \quad (4.39)$$

So, Eq.(4.36) holds on states of the type $\tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega$, which satisfy the condition (4.38). Observing that the couples $\{h_2, \tilde{a}^*(t, f_1) \cdots \tilde{a}^*(t, f_n) \Omega\}$ obeying (4.38) are norm dense in $L^2(\mathbf{R}_+) \otimes \mathcal{H}_{RB}^n$, Eq.(4.36) follows by continuity.

As a consequence of the commutation relations (4.35,36), one has the following useful estimate.

Proposition 5: Let A be a finite set and let $f_1^\alpha, \dots, f_n^\alpha \in C_0^\infty$ for any $\alpha \in A$. Then the norm of the operator

$$\sum_{\alpha \in A} \Phi(t, f_1^\alpha) \Phi(t, f_2^\alpha) \cdots \Phi(t, f_n^\alpha),$$

restricted to $\mathcal{H}_{R,B}^m$ with $m \geq n$, satisfies

$$\left\| \sum_{\alpha \in A} \Phi(t, f_1^\alpha) \Phi(t, f_2^\alpha) \cdots \Phi(t, f_n^\alpha) \right\| \leq \sqrt{m(m-1) \cdots (m-n+1)} \left\| \sum_{\alpha \in A} f_1^\alpha \otimes \cdots \otimes f_n^\alpha \right\| \quad . \quad (4.40)$$

Proof: Let $\psi \in \mathcal{D}_0^n$. Then there is some finite set B , such that ψ can be written in the form

$$\psi = \sum_{\beta \in B} \Phi^*(t, h_1^\beta) \cdots \Phi^*(t, h_m^\beta) \Omega, \quad h_1^\beta \prec \cdots \prec h_m^\beta.$$

Now, by means of the commutation relations (4.35,36), one finds

$$\left\| \sum_{\alpha \in A} \Phi(t, f_1^\alpha) \cdots \Phi(t, f_n^\alpha) \psi \right\| \leq \sqrt{m(m-1) \cdots (m-n+1)} \left\| \sum_{\alpha \in A} f_1^\alpha \otimes \cdots \otimes f_n^\alpha \right\| \|\psi\|, \quad (4.41)$$

implying Eq.(4.40) by continuity.

V. TIME EVOLUTION

In order to investigate the time evolution in the NLS model on \mathbf{R}_+ , we consider the mapping

$$\alpha_t(a(k)) = e^{-ik^2 t} a(k), \quad \alpha_t(a^*(k)) = e^{ik^2 t} a^*(k), \quad \alpha_t(b(k)) = b(k), \quad t \in \mathbf{R}. \quad (5.1)$$

It is straightforward to verify that α_t defines a 1-parameter group of automorphisms of the boundary algebra \mathcal{B}_R . Using the relations (3.2-6,13,14), one can easily check that this group is unitarily implemented in the Fock space $\mathcal{F}_{R,B}$ by means of the operator

$$U(t) = \exp(iHt), \quad H = \frac{1}{2} \int_{\mathbf{R}} \frac{dk}{2\pi} k^2 a^*(k) a(k). \quad (5.2)$$

The Hamiltonian H acts on \mathcal{D} according to

$$[H\varphi]^{(n)}(k_1, \dots, k_n) = (k_1^2 + \dots + k_n^2) \varphi^{(n)}(k_1, \dots, k_n) \quad , \quad (5.3)$$

which implies that the domain \mathcal{D} is invariant both under $U(t)$ and H . Moreover, since

$$-i \frac{d}{dt} U(t)|_{t=0} = H \quad , \quad (5.4)$$

on \mathcal{D} , the latter is a domain of essential self-adjointness for H .

The crucial point now is that the time evolution of the field $\Phi(t, f)$ is given by

$$\Phi(t, f) = U(t) \Phi(0, f) U(t)^{-1} \quad . \quad (5.5)$$

This fact follows directly from the time dependence encoded in Eqs.(4.1,2) and is quite remarkable. It shows the power of both the quantum inverse scattering transform (4.2) and the algebra \mathcal{B}_R , which combined together allow to write down the Hamiltonian of an interacting field theory as a simple quadratic expression in a and a^* . In this form H depends only implicitly on the coupling constant g through the exchange factor R . Notice also that the boundary generator b does not evolve in time.

A. The quantum equation of motion

A preliminary problem to be faced here is to give a precise meaning on the quantum level of the cubic term $|\Phi(t, x)|^2 \Phi(t, x)$ present in Eq.(1.1). For this purpose we will

follow the standard approach, introducing the concept of a normal ordered $: \dots :$ product involving Φ and Φ^* . As usually assumed, in such a product all creation operators a^* stand to the left of all annihilation operators a . In view of Eqs.(3.2,3), in our case one must further specify the ordering of creators and annihilators themselves. We define $: \dots :$ to preserve the original order of the creators. The original order of two annihilators is preserved if both belong to the same Φ or Φ^* and inverted otherwise. The quantum version of Eq.(1.1) is then obtained by the substitution

$$|\Phi(t, x)|^2 \Phi(t, x) \mapsto : \Phi \Phi^* \Phi : (t, x) \quad . \quad (5.6)$$

Concerning the relation between the above way of defining the normal product and the alternative point-splitting procedure, we observe that

$$: \Phi \Phi^* \Phi : (t, x) = \lim_{\sigma \downarrow 0} \Phi(t, x + 2\sigma) \Phi^*(t, x + \sigma) \Phi(t, x) \quad , \quad (5.7)$$

holds in mean value on \mathcal{D} . Following Ref. 6, Eq.(5.7) can be derived by using the analyticity properties of the commutator between $a(p)$ and $\Phi(t, x)$. One can formulate at this point

Theorem 5: *The Nonlinear Schrödinger equation*

$$(i\partial_t + \partial_x^2) \langle \varphi, \Phi(t, x) \psi \rangle = 2g \langle \varphi, : \Phi \Phi^* \Phi : (t, x) \psi \rangle \quad , \quad (5.8)$$

is satisfied for any $\varphi, \psi \in \mathcal{D}$.

Proof: The first step is analogous to the proof of Proposition 2 and consists in showing that the matrix element $\langle \varphi, : \Phi \Phi^* \Phi : (t, x) \psi \rangle$ is smooth in t and x for any $\varphi, \psi \in \mathcal{D}$. The next step is to compare $(i\partial_t + \partial_x^2) \langle \varphi, \Phi^{(n)}(t, x) \psi \rangle$ with the $(n-1)$ -th order term in the expansion of $\langle \varphi, : \Phi \Phi^* \Phi : (t, x) \psi \rangle$ in terms of g . A straightforward computation, similar to that performed in Ref. 8 for the NLS model on \mathbf{R} , shows that these terms indeed coincide.

B. Boundary conditions

We shall demonstrate now

Theorem 6. *The following boundary conditions hold for any $\varphi, \psi \in \mathcal{D}$ and $t \in \mathbf{R}$:*

$$\lim_{x \downarrow 0} (\partial_x - \eta) \langle \varphi, \Phi(t, x) \psi \rangle = 0, \quad (5.9)$$

$$\lim_{x \rightarrow \infty} \langle \varphi, \Phi(t, x) \psi \rangle = 0. \quad (5.10)$$

Let us first prove

Lemma 3: *Let $\varphi, \psi \in \mathcal{F}_{R,B}^0$. There exists a vector $\chi \in \mathcal{H}_{R,B}^1$ such that*

$$\langle \varphi, \Phi(t, f) \psi \rangle = \langle \Omega, \Phi(t, f) \chi \rangle. \quad (5.11)$$

Proof: Without loss of generality one can take $\varphi \in \mathcal{H}_{R,B}^n$, $\psi \in \mathcal{H}_{R,B}^{n+1}$. Suppose first that $\varphi \in \mathcal{E}_0^n = \mathcal{D}_0^n$. Then φ is of the form

$$\varphi = \sum_{\alpha \in A} \Phi^*(t, f_1^\alpha) \Phi^*(t, f_2^\alpha) \cdots \Phi^*(t, f_n^\alpha) \Omega, \quad (5.12)$$

where A is a finite set and $f_1^\alpha \prec f_2^\alpha \prec \cdots \prec f_n^\alpha$ for all $\alpha \in A$. Using the commutation relations (4.35,36), one easily obtains

$$\langle \varphi, \Phi(t, f) \psi \rangle = \sum_{\alpha \in A} \langle \Omega, \Phi(t, f) \Phi(t, f_n^\alpha) \Phi(t, f_{n-1}^\alpha) \cdots \Phi(t, f_1^\alpha) \psi \rangle. \quad (5.13)$$

In order to solve (5.11), it is then sufficient to define

$$\chi = \Phi(t, f_n^\alpha) \Phi(t, f_{n-1}^\alpha) \cdots \Phi(t, f_1^\alpha) \psi, \quad (5.14)$$

which belongs to $\mathcal{H}_{R,B}^1$ since $\psi \in \mathcal{H}_{R,B}^{n+1}$. Take now a general $\varphi \in \mathcal{H}_{R,B}^n$. By cyclicity (Theorem 3), there exists a sequence $\{\varphi_k\} \subset \mathcal{D}_0^n$ converging to φ . By Proposition 5, the corresponding vectors $\{\chi_k\}$ given by Eq.(5.14) form a Cauchy sequence, which converges to a vector $\chi \in \mathcal{H}_{R,B}^1$, satisfying (5.11) by continuity.

We can now prove Theorem 6.

Proof: Let $\varphi, \psi \in \mathcal{D}_0 \subset \mathcal{F}_{R,B}^0$. From the lemma above there exists $\chi \in \mathcal{H}_{R,B}^1$ such that

$$\langle \varphi, \Phi(t, x) \psi \rangle = \langle \Omega, \Phi(t, x) \chi \rangle = \int_{\mathbf{R}} \frac{dk}{2\pi} e^{ikx - ik^2 t} \chi(k). \quad (5.15)$$

Since $\chi \in L^2$, the matrix element $\langle \varphi, \Phi(t, x)\psi \rangle$, which by Proposition 2 is smooth, is also square integrable with respect to x . Therefore it vanishes at infinity and Eq.(5.10) is satisfied. Moreover, taking the derivative with respect to x , the B -symmetry (3.16) of χ , immediately leads to Eq.(5.9).

C. Correlation Functions

From the general structure of our solution it follows that:

- (i) the nonvanishing correlation functions involve equal number of Φ and Φ^* ;
- (ii) for computing the exact $2n$ -point function one does not need all terms in the expansion (2.3), but at most the $(n - 1)$ -th order contribution.

One has for instance:

$$\langle \Omega, \Phi(t_1, x_1)\Phi^*(t_2, x_2)\Omega \rangle = \langle \Omega, \Phi^{(0)}(t_1, x_1)\Phi^{*(0)}(t_2, x_2)\Omega \rangle, \quad (5.16)$$

$$\begin{aligned} & \langle \Omega, \Phi(t_1, x_1)\Phi(t_2, x_2)\Phi^*(t_3, x_3)\Phi^*(t_4, x_4)\Omega \rangle = \\ & \langle \Omega, \Phi^{(0)}(t_1, x_1)\Phi^{(0)}(t_2, x_2)\Phi^{*(0)}(t_3, x_3)\Phi^{*(0)}(t_4, x_4)\Omega \rangle + \\ & g^2 \langle \Omega, \Phi^{(0)}(t_1, x_1)\Phi^{(1)}(t_2, x_2)\Phi^{*(1)}(t_3, x_3)\Phi^{*(0)}(t_4, x_4)\Omega \rangle. \end{aligned} \quad (5.17)$$

Since the vacuum expectation value of any number of $\{a(k), a^*(k), b(k)\}$ is known explicitly¹¹, employing Eqs.(4.1,2,6,7) one can derive integral representations for the NLS correlation functions on \mathbf{R}_+ . For example,

$$\langle \Omega, \Phi(t_1, x_1)\Phi^*(t_2, x_2)\Omega \rangle = \int_{\mathbf{R}} \frac{dp}{2\pi} e^{-ip^2(t_1-t_2)} \left[e^{ip(x_1-x_2)} + B(p)e^{ip(x_1+x_2)} \right], \quad (5.18)$$

which coincides with that of the non-relativistic free field on the half line. In spite of this fact, the four-point function (5.17) differs from the free one. We would like to recall in this respect that according to Jost's theorem (see e.g. Ref. 16), such a phenomenon is forbidden in relativistic invariant models.

VI. SCATTERING THEORY

As it is well known, integrable quantum systems on the real line are characterized by a factorized scattering matrix. This means that multiparticle scattering is described

by an appropriate product of two-particle scattering matrices, which in turn are subject to physical constraints like unitarity, crossing symmetry, etc.

Some years ago, Cherednik¹² proposed a version of factorized scattering, adapted to the half line case. The following physical picture emerges from his investigation. Let $|k_1, \dots, k_n\rangle^{\text{in}}$ be an in-state, representing n particles coming from $x = +\infty$ and thus having negative momenta $k_1 < k_2 < \dots < k_n < 0$. These particles interact among themselves before and after being reflected by the wall at $x = 0$, giving rise to an out-state $|p_1, \dots, p_m\rangle^{\text{out}}$ composed of particles traveling towards $x = +\infty$ and thus having positive momenta $p_1 > p_2 > \dots > p_m > 0$. The transition amplitude between these states vanishes unless $n = m$ and $p_i = -k_i$, $i = 1, \dots, n$. Therefore, not only the total momentum, but each momentum is separately reflected. According to Ref. 12, the scattering amplitude is

$$\langle p_1, \dots, p_m | k_1, \dots, k_n \rangle^{\text{in}} = \delta_{mn} \prod_{i=1}^n 2\pi \delta(p_i + k_i) B(p_i) \prod_{\substack{i,j=1 \\ i < j}}^n R(p_i, p_j) R(p_i, -p_j) . \quad (6.1)$$

The R -factors describe the interactions among the particles in the bulk, while the B -factors take into account the reflection from the wall.

The main goal of this section is to prove that the NLS model on \mathbf{R}_+ perfectly fits the scheme of Cherednik. In order to do that, we must develop first the scattering theory corresponding to the off-shell quantum field $\Phi^*(t, f)$. Our framework will be the conventional Haag-Ruelle approach¹⁷, suitably adapted to the nonrelativistic case.

A first relation between the quantum solution (4.6,7) and Cherednik's scattering amplitude (6.1) is obtained through the identification

$$|p_1, \dots, p_n\rangle^{\text{out}} = a^*(p_1) \dots a^*(p_n) \Omega , \quad p_1 > \dots > p_n > 0 , \quad (6.2)$$

$$|k_1, \dots, k_n\rangle^{\text{in}} = a^*(k_1) \dots a^*(k_n) \Omega , \quad k_1 < \dots < k_n < 0 . \quad (6.3)$$

We recall in fact that \mathcal{B}_R has been designed in such a way, that the amplitudes

$$\langle a^*(p_1) \dots a^*(p_m) \Omega , a^*(k_1) \dots a^*(k_n) \Omega \rangle , \quad (6.4)$$

precisely reproduce the right hand side of Eq.(6.1). What is still missing therefore is the construction of suitable states, expressed in terms of $\Phi^*(t, h)$ and Ω , which approach

the out-states (6.2) for $t \rightarrow \infty$ and the in-states (6.3) for $t \rightarrow -\infty$. We are now going to fill this gap.

Proposition 5 shows that $\Phi^*(t, f)$, restricted on $\mathcal{H}_{R,B}^n$ is a bounded operator of norm

$$\|\Phi^*(t, f)\| \leq \sqrt{n+1} \|f\|, \quad (6.5)$$

which in turn implies that it can be extended to any $f \in L^2(\mathbf{R}_+)$. From the estimates (3.21) we know that also $a^*(h)$ is bounded on $\mathcal{H}_{R,B}^n$, where

$$\|a^*(h)\| \leq \sqrt{n+1} \|h\|, \quad \forall h \in L^2(\mathbf{R}). \quad (6.6)$$

Combining this inequality with the definition (4.6), one finds,

$$\left\| \int_{\mathbf{R}^n} dx_1 \dots dx_n f(x_1, \dots, x_n) \tilde{a}^*(t, x_1) \cdots \tilde{a}^*(t, x_n) \Omega \right\| \leq \sqrt{n!} \|f\|, \quad \forall f \in L^2(\mathbf{R}^n). \quad (6.7)$$

In order to develop the Haag-Ruelle formalism, we will need also the following notations. Let $h(k) \in \mathcal{S}(\mathbf{R})$. Then we set:

$$h^t(x) \equiv \int_{\mathbf{R}} \frac{dk}{2\pi} e^{ikx - ik^2 t} h(k), \quad h_+^t(x) \equiv \theta(x)[h^t(x) + h^t(-x)], \quad \tilde{h}(k) \equiv h(-k), \quad (6.8)$$

where $\theta(x)$ is the Heaviside step function. Notice that

$$\tilde{h}_+^t(x) = \theta(x)[\tilde{h}^t(x) + \tilde{h}^t(-x)] = \theta(x)[h^t(-x) + h^t(x)] = h_+^t(x). \quad (6.9)$$

We are now in position to formulate

Theorem 7: (Asymptotic states) *Let*

$$h_1 \succ h_2 \succ \cdots \succ h_n, \quad h_j \in \mathcal{S}(\mathbf{R}_+), \quad j = 1, \dots, n.$$

Then one has the following strong limits

$$\lim_{t \rightarrow +\infty} \Phi^*(t, h_{1+}^t) \Phi^*(t, h_{2+}^t) \cdots \Phi^*(t, h_{n+}^t) \Omega = a^*(h_1) a^*(h_2) \cdots a^*(h_n) \Omega, \quad (6.10)$$

$$\lim_{t \rightarrow -\infty} \Phi^*(t, h_{1+}^t) \Phi^*(t, h_{2+}^t) \cdots \Phi^*(t, h_{n+}^t) \Omega = a^*(\tilde{h}_1) a^*(\tilde{h}_2) \cdots a^*(\tilde{h}_n) \Omega. \quad (6.11)$$

For proving this statement, we need some preliminary results.

Lemma 4: Let $h \in \mathcal{S}(\mathbf{R}_+)$. Then

$$\lim_{t \rightarrow +\infty} \|h_+^t - h^t\| = 0, \quad \lim_{t \rightarrow -\infty} \|h_+^t - \tilde{h}^t\| = 0. \quad (6.12)$$

Proof: A direct computation gives:

$$\|h_+^t - h^t\|^2 = 2i \int_{\mathbf{R}^2} \frac{dk}{2\pi} \frac{dp}{2\pi} \bar{h}(k) h(p) \frac{e^{it(k+p)(k-p)}}{k-p+i\epsilon}, \quad (6.13)$$

$$\|h_+^t - \tilde{h}^t\|^2 = -2i \int_{\mathbf{R}^2} \frac{dk}{2\pi} \frac{dp}{2\pi} \bar{h}(k) h(p) \frac{e^{it(k+p)(k-p)}}{k-p-i\epsilon}. \quad (6.14)$$

Now, for proving Eq.(6.12), it is enough to take into account that $\text{supp} h > 0$ and to use the weak limit

$$\lim_{t \rightarrow \pm\infty} \frac{e^{itk}}{k \pm i\epsilon} = 0. \quad (6.15)$$

Corollary 1: Let $h_1, h_2, \dots, h_n \in \mathcal{S}(\mathbf{R}_+)$. Then

$$\lim_{t \rightarrow +\infty} \|h_{1+}^t \otimes \dots \otimes h_{n+}^t - h_1^t \otimes \dots \otimes h_n^t\| = 0, \quad (6.16)$$

$$\lim_{t \rightarrow -\infty} \|h_{1+}^t \otimes \dots \otimes h_{n+}^t - \tilde{h}_1^t \otimes \dots \otimes \tilde{h}_n^t\| = 0. \quad (6.17)$$

Lemma 5: Let $h_1, h_2 \in \mathcal{S}(\mathbf{R}_+)$ are such that $h_1 \succ h_2$. Then, the functions

$$H^t(x_1, x_2) = h_1^t(x_1) h_2^t(x_2) \theta(x_2 - x_1), \quad (6.18)$$

$$\tilde{H}^t(x_1, x_2) = \tilde{h}_1^t(x_1) \tilde{h}_2^t(x_2) \theta(x_2 - x_1), \quad (6.19)$$

satisfy

$$\lim_{t \rightarrow +\infty} \|H^t\| = 0, \quad \lim_{t \rightarrow -\infty} \|\tilde{H}^t\| = 0. \quad (6.20)$$

Proof: Let us consider for instance H^t . One has

$$\begin{aligned} \|H^t\|^2 &= \int_{-\infty}^{\infty} dx_1 \int_{x_1}^{\infty} dx_2 |h_1^t(x_1) h_2^t(x_2)|^2 = \\ &= \int_{\mathbf{R}^4} \frac{dk_1}{2\pi} \frac{dp_1}{2\pi} \frac{dk_2}{2\pi} \frac{dp_2}{2\pi} \bar{h}_1(k_1) h_1(p_1) \bar{h}_2(k_2) h_2(p_2) I(k_1, p_1, k_2, p_2) e^{(k_1^2 - p_1^2 + k_2^2 - p_2^2)t}, \end{aligned} \quad (6.21)$$

with

$$I(k_1, p_1, k_2, p_2) \equiv \int_{-\infty}^{\infty} dx_1 \int_{x_1}^{\infty} dx_2 e^{i[(p_1 - k_1)x_1 + (p_2 - k_2)x_2]}.$$

The integration in x_1 and x_2 gives

$$I(k_1, p_1, k_2, p_2) = \frac{2\pi i \delta(k_1 - p_1 + k_2 - p_2)}{p_2 - k_2 + i\epsilon}. \quad (6.22)$$

Therefore,

$$\|H^t\|^2 = i \int_{\mathbf{R}^3} \frac{dp_1}{2\pi} \frac{dp_2}{2\pi} \frac{dk_2}{2\pi} \bar{h}_1(p_1 - p_2 + k_2) h_1(p_1) \bar{h}_2(k_2) h_2(p_2) \frac{e^{2i(p_1 - k_2)(p_2 - k_2)t}}{p_2 - k_2 + i\epsilon}. \quad (6.23)$$

The support properties of the function h_1 and h_2 imply that the integrand vanishes unless $p_1 > k_2 > 0$, which completes the argument because of Eq.(6.15). Analogous considerations apply to \tilde{H}^t .

Corollary 2: Let

$$G^t(x_1, x_2) = h_{+1}^t(x_1) h_{+2}^t(x_2) \theta(x_2 - x_1). \quad (6.24)$$

Then

$$\lim_{t \rightarrow \pm\infty} \|G^t\| = 0. \quad (6.25)$$

Proof: One has to combine Eqs.(6.9,16,17,20).

The statement of Corollary 2 has the following generalization to the case of $n \geq 2$ variables. Suppose that $h_1, \dots, h_n \in \mathcal{S}(\mathbf{R}_+)$ and $h_1 \succ \dots \succ h_n$. Let \mathcal{P}_n be the group of all permutations of the indices $\{1, 2, \dots, n\}$. For any $\sigma \in \mathcal{P}_n$ we define the function

$$G_\sigma^t(x_1, \dots, x_n) \equiv h_{1+}^t(x_1) \cdots h_{n+}^t(x_n) \theta(x_{\sigma_1}, \dots, x_{\sigma_n}), \quad (6.26)$$

where

$$\theta(x_{\sigma_1}, \dots, x_{\sigma_n}) \equiv \prod_{\substack{i,j=1 \\ i < j}}^n \theta(x_{\sigma_i} - x_{\sigma_j}). \quad (6.27)$$

Corollary 3: For any $\sigma \in \mathcal{P}_n$ different from the identity $e = (1, 2, \dots, n)$, one has

$$\lim_{t \rightarrow \pm\infty} \|G_\sigma^t\| = 0. \quad (6.28)$$

We are now ready to prove Theorem 7.

Proof: The case $n = 1$ is quite simple. Using the identities

$$\Phi^*(t, f)\Omega = \tilde{a}^*(t, f)\Omega, \quad a^*(h) = \tilde{a}^*(t, h^t) \quad , \quad (6.29)$$

one finds

$$\|\Phi^*(t, h_+^t)\Omega - a^*(h)\Omega\| = \|\tilde{a}^*(t, h_+^t)\Omega - \tilde{a}^*(t, h^t)\Omega\| \leq \|h_+^t - h^t\|, \quad (6.30)$$

which according to Lemma 5 tends to 0 in the limit $t \rightarrow +\infty$. Let us consider now the case $n \geq 2$. Applying Eq.(4.34) and

$$\theta(x_1, \dots, x_n) = 1 - \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma \neq e}} \theta(x_{\sigma_1}, \dots, x_{\sigma_n}), \quad (6.31)$$

we get:

$$\begin{aligned} & \Phi^*(t, h_{1+}^t) \cdots \Phi^*(t, h_{n+}^t)\Omega = \\ & \int_{\mathbf{R}^n} dx_1 \dots dx_n h_{1+}^t(x_1) \cdots h_{n+}^t(x_n) \sum_{\sigma \in \mathcal{P}_n} \theta(x_{\sigma_1}, \dots, x_{\sigma_n}) \tilde{a}^*(t, x_{\sigma_1}) \cdots \tilde{a}^*(t, x_{\sigma_n})\Omega = \\ & \tilde{a}^*(t, h_{1+}^t) \cdots \tilde{a}^*(t, h_{n+}^t)\Omega + \\ & \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma \neq e}} \int_{\mathbf{R}^n} dx_1 \dots dx_n G_\sigma^t(x_1, \dots, x_n) [\tilde{a}^*(t, x_{\sigma_1}) \cdots \tilde{a}^*(t, x_{\sigma_n})\Omega - \tilde{a}^*(t, x_1) \cdots \tilde{a}^*(t, x_n)\Omega]. \end{aligned} \quad (6.32)$$

The estimate (6.7) then leads to

$$\begin{aligned} & \|\Phi^*(t, h_{1+}^t) \cdots \Phi^*(t, h_{n+}^t)\Omega - a^*(h_1) \cdots a^*(h_n)\Omega\| = \\ & \|\Phi^*(t, h_{1+}^t) \cdots \Phi^*(t, h_{n+}^t)\Omega - \tilde{a}^*(t, h_1^t) \cdots \tilde{a}^*(t, h_n^t)\Omega\| \leq \\ & \|\tilde{a}^*(t, h_{1+}^t) \cdots \tilde{a}^*(t, h_{n+}^t)\Omega - \tilde{a}^*(t, h_1^t) \cdots \tilde{a}^*(t, h_n^t)\Omega\| + 2\sqrt{n!} \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma \neq e}} \|G_\sigma^t\| \leq \\ & \sqrt{n!} \|h_{1+}^t \otimes \cdots \otimes h_{n+}^t - h_1^t \otimes \cdots \otimes h_n^t\| + 2\sqrt{n!} \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma \neq e}} \|G_\sigma^t\|, \end{aligned} \quad (6.33)$$

which implies the strong limit (6.10). Analogous considerations give

$$\begin{aligned} & \|\Phi^*(t, h_{1+}^t) \cdots \Phi^*(t, h_{n+}^t)\Omega - a^*(\tilde{h}_1) \cdots a^*(\tilde{h}_n)\Omega\| \leq \\ & \sqrt{n!} \|h_{1+}^t \otimes \cdots \otimes h_{n+}^t - \tilde{h}_1^t \otimes \cdots \otimes \tilde{h}_n^t\| + 2\sqrt{n!} \sum_{\substack{\sigma \in \mathcal{P}_n \\ \sigma \neq e}} \|G_\sigma^t\|, \end{aligned} \quad (6.34)$$

which proves (6.11).

We proceed with the construction of the scattering operator S , following the general strategy developed in Ref. 18. According to Theorem 7, the asymptotic spaces \mathcal{F}^{out} and \mathcal{F}^{in} are generated by finite linear combinations of the vectors ($n \geq 1$)

$$\mathcal{E}^{\text{out}} = \{ \Omega, a^*(h_1) \cdots a^*(h_n) \Omega : h_1 \succ \cdots \succ h_n, h_j \in \mathcal{S}(\mathbf{R}_+) \} \quad (6.35)$$

and

$$\mathcal{E}^{\text{in}} = \{ \Omega, a^*(\tilde{h}_1) \cdots a^*(\tilde{h}_n) \Omega : h_1 \succ \cdots \succ h_n, h_j \in \mathcal{S}(\mathbf{R}_+) \} \quad (6.36)$$

respectively. One can show¹¹ moreover, that \mathcal{F}^{out} and \mathcal{F}^{in} are separately dense $\mathcal{F}_{R,B}$. This property of asymptotic completeness allows to demonstrate¹¹ that the mapping $S : \mathcal{E}^{\text{out}} \rightarrow \mathcal{E}^{\text{in}}$, defined by

$$S\Omega = \Omega \quad , \quad (6.37)$$

$$S a^*(h_1) a^*(h_2) \cdots a^*(h_n) \Omega = a^*(\tilde{h}_1) a^*(\tilde{h}_2) \cdots a^*(\tilde{h}_n) \Omega \quad , \quad (6.38)$$

extends to a unitary scattering operator on $\mathcal{F}_{R,B}$. We stress that S is nontrivial, in spite of the fact that the quantum fields Φ and Φ^* realize a Fock representation of the canonical commutation relations. This feature is not in contradiction with Haag's theorem¹⁶, because we are dealing with a nonrelativistic system, which does not satisfy in particular relativistic local commutativity.

The construction of the scattering operator S completes the picture and concludes our quantum field theory description of the NLS model on \mathbf{R}_+ .

VII. OUTLOOK AND CONCLUSIONS

We studied the nonlinear Schrödinger equation on the half line with mixed boundary condition. After a brief discussion of some aspects of the corresponding classical boundary value problem, we constructed the exact second quantized solution of the system, establishing its basic properties. The explicit form of our solution shows that the quantum inverse scattering transform works also on the half line, provided that the Zamolodchikov-Faddeev algebra is replaced by the boundary algebra \mathcal{B}_R . This is one of the main results of the present paper. It demonstrates that besides being an useful tool in scattering theory¹¹, the concept of boundary algebra is essential also for the

construction of off-shell interacting fields in integrable systems on \mathbf{R}_+ . We emphasize in this respect, that our results have a straightforward generalization to all elements of the NLS hierarchy (e.g. the complex modified Korteweg-de Vries equation) on the half line. The case with internal $SU(N)$ symmetry can also be treated analogously.

As for future extensions of the present work, it would be interesting to investigate the range $\eta < 0$. The new phenomenon, which can be expected on general grounds, is the presence of boundary bound states. Taking into account that one can describe by \mathcal{B}_R also degrees of freedom residing on the boundary (see the appendix of Ref. 11), we strongly believe that our framework extends to the case $\eta < 0$ as well.

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